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# EQUILIBRIUM OF A FREE NONISOTHERMAL LIQUID FILM

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*The equilibrium of a free weightless liquid film fixed over a planar contour and acted upon by thermocapillary forces is studied. Trends in the behavior of free liquid films are important for understanding the processes occurring in foams. The equilibrium equations for a nonisothermal weightless free film are derived for the two limiting cases: the temperature of the film is considered a known function of the coordinates; the free surface of the film is thermally insulated. For the plane and axisymmetric cases, the existence conditions for the solutions of the resulting nonlinear boundary-value problems are found and their properties are studied. For the general case, an approximate solution of the equilibrium problem is obtained provided that the analogue of the Marangoni number is small.*

**Key words:** *free surface, thermocapillary effect, long-wave approximation, stationary solutions.*

**1. Formulation of the Problem.** Let a viscous incompressible liquid fill a layer $\Omega$ whose upper and lower boundaries $\Gamma^+$ and $\Gamma^-$ are free and whose lateral surface is adjacent to a solid cylindrical surface $\Sigma$ with generatrices parallel to the $x_3$ axis. Below, the following notation is used: $x_1$, $x_2$, and $x_3$ are Cartesian coordinates, $v_1$, $v_2$, and $v_3$ are the corresponding velocity components, and $p$ is the liquid pressure. The liquid density $\rho$, the kinematic viscosity $\nu$, and the thermal diffusivity $\chi$ are considered constant, and the surface-tension coefficient $\sigma$ is assumed to be a linear function of the temperature $T$:

$$\sigma = \sigma_0 - \varkappa(T - T_0) \tag{1.1}$$

($\sigma_0$, $\varkappa$, and $T_0$ are positive constants). Next, it is assumed that the liquid flow is stationary and symmetric about the plane $x_3 = 0$. In addition, it is assumed that surface-active substances and external mass forces are absent.

The mathematical problem consists of determining the region $\Omega$ and the solution of the system of the Navier–Stokes and thermal-conductivity equations

$$\boldsymbol{v} \cdot \nabla_3 \boldsymbol{v} = -\rho^{-1} \nabla_3 p + \nu \Delta_3 \boldsymbol{v}, \qquad \nabla_3 \cdot \boldsymbol{v} = 0; \tag{1.2}$$

$$\boldsymbol{v} \cdot \nabla_3 T = \chi \Delta_3 T \tag{1.3}$$

in this region subject to the free-boundary conditions

$$-p\boldsymbol{N} + 2\rho\nu D \cdot \boldsymbol{N} = -2K\sigma\boldsymbol{N} + \nabla_\Gamma \sigma; \tag{1.4}$$

$$\boldsymbol{v} \cdot \boldsymbol{N} = 0, \qquad x \in \Gamma^\pm, \tag{1.5}$$

the no slip condition on the solid part of the boundary

$$\boldsymbol{v} = 0, \qquad x \in \Sigma, \tag{1.6}$$

the thermal-contact conditions formulated below, and symmetry conditions. The latter imply that the surface $\Gamma^-$ is a reflection of $\Gamma^+$ with respect to the plane $x_1$, $x_2$; in addition, the functions $v_1$, $v_2$, $p$, and $T$ are even functions of the variable $x_3$, and $v_3$ is an odd function of $x_3$.

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In relations (1.2)–(1.5), $\nabla_3$ and $\Delta_3$ are the three-dimensional gradient and Laplacian, respectively, $D = [\nabla_3 \boldsymbol{v} + (\nabla_3 \boldsymbol{v})^*]/2$ is the strain rate tensor, $\boldsymbol{N}$ is the unit outward normal vector to the surface $\Gamma^+$, $K$ is the average curvature of this surface; and $\nabla_\Gamma = \nabla - \boldsymbol{N}(\boldsymbol{N} \cdot \nabla\,)$ is the surface gradient.

If the quantity $\sigma$ in condition (1.4) is constant, the dynamic problem (1.2), (1.4)–(1.6) is separated from the thermal one and has a solution in which $p = \text{const}$, $\boldsymbol{v} = 0$, and $\Gamma^+$ is defined as the surface of specified constant average curvature for a specified value of the contact angle. If $\sigma \neq \text{const}$ [which is inevitable at variable temperature, by virtue of equality (1.1)], the problem is considerably complicated. Since the liquid is in contact with the solid body along the surface $\Sigma$ and with the gas phase at the points of the free boundary, additional boundary conditions should be specified on the surfaces $\Sigma$ and $\Gamma^+$. We assume that the temperature or heat-flux distribution on the surface $\Sigma$ is known:

$$T = f(x) \qquad (x \in \Sigma) \tag{1.7}$$

or

$$\frac{\partial T}{\partial n} = q(x) \qquad (x \in \Sigma). \tag{1.8}$$

Here $f(x)$ and $q(x)$ are specified functions and $\partial/\partial n$ is the derivative with respect to the direction of the outward normal $\boldsymbol{n}$ to the surface $\Sigma$. As regards the condition of thermal contact of the film with the gas phase, it is usually formulated as a condition of the 3rd kind for the temperature that includes an empirical constant (the interphase heat exchange coefficient). It does not need to be determined in the two limiting cases: a thermally insulated free boundary and ideal thermal contact of the liquid and gas phases. In the latter case, we assume that the free-surface temperature $\theta$ is a specified function of the coordinates $x_1$ and $x_2$. After the substitution of equalities (1.1) and $T = \theta$ for $x \in \Gamma^+$ into condition (1.4), the problem of determining the functions $\boldsymbol{v}$ and $p$ and the surface $\Gamma^+$ becomes closed and the function $T$ in the region $\Omega$ is determined *a posteriori*.

If the free surface is thermally insulated, the condition for the temperature is written as

$$\frac{\partial T}{\partial N} = 0 \qquad (x \in \Gamma^+), \tag{1.9}$$

where $\partial/\partial N$ is the derivative with respect to the direction of the outward normal to the surface $\Gamma^+$. In this case, if the problem is closed by condition (1.8), the function $q(x)$ entering this condition should be subjected to the relation

$$\int\limits_\Sigma q\, d\Sigma = 0. \tag{1.10}$$

In addition, it is necessary to specify the three-phase contact angle at the points of intersection of the surfaces $\Sigma$ and $\Gamma^+$. We confine ourselves to the simple case where this angle is equal to $\pi/2$. In this case, the following the relation holds:

$$\frac{\partial h}{\partial n} = 0, \qquad x \in \Sigma. \tag{1.11}$$

Here $h$ is a function that specifies the free surface by means of the equality $x_3 = h(x_1, x_2)$. Finally, for the solution to be uniquely determined, the volume of the region occupied by the liquid needs to be specified:

$$\int\limits_\omega h(x_1, x_2)\, dx_1\, dx_2 = Q \tag{1.12}$$

($\omega$ is the section of the region $\Omega$ by the plane $x_3 = 0$).

Thus, two problems with the unknown boundary are formulated for system (1.2), (1.3). In the first of these, the boundary conditions have the form (1.4)–(1.7), (1.11), (1.12), and the values of the function $T$ on the surface $\Gamma^+$ [only these values appear in condition (1.4)] are specified *a priori*:

$$T = \theta(x_1, x_2) \qquad (x \in \Gamma^+). \tag{1.13}$$

Problem (1.1)–(1.7), (1.11)–(1.13) will be called problem A. In the second problem, the set of boundary conditions is replaced by (1.4)–(1.6) and (1.8)–(1.12). This problem for system (1.2), (1.3) will be called problem B.

Generally, problems A and B can be solved only numerically. However, in the case where the film thickness is much smaller than the diameter of the region $\omega$ and the derivatives of the required functions with respect to the transverse coordinate $x_3$ are much larger than their derivatives with respect to the longitudinal coordinates $x_1$

and $x_2$, one can use the thin-layer approximation [1]. This simplifies the problem considerably but does not make it trivial. The main simplification provided by this approximation is that the problem with the unknown boundary becomes a problem in a fixed region.

**2. Thin-Layer Approximation.** We denote by $l$ the diameter of the plane region $\omega$ and assume that for $(x_1, x_2) \in \omega$, the relations $h = \varepsilon l$ and $|\nabla h| = O(\varepsilon)$ and $l\Delta h = O(\varepsilon)$ are satisfied when $\varepsilon \to 0$ (here and below, $\nabla$ and $\Delta$ are the gradient and Laplacian over the variables $x_1$ and $x_2$). We denote by $\delta T$ the characteristic temperature drop along the film and assume that the change in the surface-tension coefficient, which has order $\varkappa\delta T$, is much smaller than its average value $\sigma_0$ (this assumption is always valid for real thermocapillary flows). We assume that $\varkappa\,\delta T/\sigma_0 = O(\varepsilon^2)$ as $\varepsilon \to 0$.

The problems considered have two characteristic linear scales: longitudinal and transverse. Accordingly, there are two velocity scales, the transverse scale being much smaller than the longitudinal scale $v_3(v_1^2 + v_2^2)^{-1/2} = O(\varepsilon)$. The characteristic longitudinal velocity $V$ can be estimated from the balance of the tangential stresses on the free boundary by virtue of relations (1.4) and (1.1), whence follows $V = \varepsilon\varkappa\,\delta T/(\rho\nu)$. From the balance of the normal stresses, the characteristic pressure is expressed as $\bar{p} = \varepsilon\sigma_0/l$. The natural length scale is the quantity $l$.

In relations (1.2)–(1.13), we transform to dimensionless variables using the formulas

$$x_i' = \frac{x}{l}, \quad v_i' = \frac{v_i}{V} \quad (i = 1, 2), \quad x_3' = \frac{x_3}{\varepsilon l}, \quad v_3' = \frac{v_3}{\varepsilon V}, \quad p' = \frac{p}{\bar{p}}, \quad h' = \frac{h}{\varepsilon l}, \quad T' = \frac{T}{\delta T}.$$

Below, the primes at the dimensionless variables are omitted. We assume that the required dimensionless functions and their derivatives with respect to the dimensionless coordinates of the order of unity for $\varepsilon \to 0$. In [1], asymptotic simplification of the equations and initial and boundary conditions of the nonstationary analog of problem A was performed based on the assumption of the existence of the finite positive limits

$$\frac{\varkappa\,\delta T}{\varepsilon^2\sigma_0} \to \gamma, \quad \frac{(\varkappa\,\delta T)^2}{\varepsilon\rho\nu^2\sigma_0} \to \beta \quad \text{at} \quad \varepsilon \to 0.$$

As a result, an equation for the film thickness was derived, which in the stationary case becomes

$$\nabla\cdot(h\nabla\Delta h) = \gamma\Delta\theta. \tag{2.1}$$

Equation (2.1) should be solved in the region $\omega \in \mathbb{R}^2$ subject to the following conditions on the boundary $\partial\omega$ of the region $\omega$:

$$\frac{\partial h}{\partial n} = 0, \qquad h\,\frac{\partial\Delta h}{\partial n} = \gamma\,\frac{\partial\theta}{\partial n}, \qquad (x_1, x_2) \in \partial\omega. \tag{2.2}$$

The first of these is condition (1.11) written in the new variables. The second condition (2.2) is derived in [1]. This condition follows from the impermeability of the surface $\Sigma$. Relations (2.1) and (2.2) are supplemented by the condition for the dimensionless liquid volume:

$$\int\limits_{\omega} h(x_1, x_2)\,d\omega = S \tag{2.3}$$

($S$ is the surface area of the region $\omega$). It is equivalent to condition (1.12) if the ratio of the dimensional volume $Q$ to the cross-sectional area of the film is chosen as the small parameter $\varepsilon$.

A remarkable feature of the problem considered is that the shape of the free surface of the film is determined by solving problem (2.1)–(2.3) in the lack of detailed information on the dependence of the velocity on the vertical coordinate $x_3$. This differs the problem of the motion of a free film under the action of thermocapillary forces differs from the classical problem of the motion of a thin viscous liquid layer adjacent to a solid plane. If the function $h(x_1, x_2)$ is determined, the velocity field in the film is found by solving the boundary-value problem in a fixed region formulated in [1].

Let us formulate problem B in the thin-layer approximation. In this problem, the temperature of the liquid $T$ is not specified, but, by virtue of condition (1.9), its dependence on the vertical coordinate is weak and, to within terms of order $\varepsilon$, it can be ignored. We make the additional assumption of smallness of the Peclet number $\mathrm{Pe} = Vl/\chi$, which allows Eq. (1.3) to be reduced to the Laplace equation $\Delta_3 T = 0$ by ignoring the nonlinear term in this equation. Setting $T = T(x_1, x_2)$ and integrating the last equation over the variable $x_3$ from zero to $h(x_1, x_2)$ subject to condition (1.9) and the symmetry conditions, we obtain the equality

$$\nabla\cdot(h\nabla T) = 0. \tag{2.4}$$

We note that the assumption $\mathrm{Pe} \ll 1$ is equivalent to the condition of smallness of the Reynolds number for liquid film motion if the Prandtl number for the liquid is of the order of unity. The latter condition underlies many approximate models of film flows.

The boundary condition for Eq. (2.4) follows from condition (1.8) in which the function $g$ can be considered independent of the variable $x_3$. Then,

$$h\,\frac{\partial T}{\partial n} = q(x), \qquad (x_1, x_2) \in \partial\omega. \tag{2.5}$$

The necessary solvability condition for the Neumann type problem (2.4), (2.5) is specified by the equality

$$\int\limits_{\partial\omega} q\,ds = 0, \tag{2.6}$$

where $ds$ is a length element of the curve $\partial\omega$.

The second equation of the system for the functions $h$ and $T$ has the form of (2.1) in which the right side contains $\Delta T$ instead of the known function $\Delta\theta$:

$$\nabla \cdot (h\nabla\Delta h) = \gamma\Delta T. \tag{2.7}$$

The boundary conditions for (2.7) are similar to (2.2):

$$\frac{\partial h}{\partial n} = 0, \qquad h\,\frac{\partial \Delta h}{\partial n} = \gamma\,\frac{\partial T}{\partial n}, \qquad (x_1, x_2) \in \partial\omega. \tag{2.8}$$

Finally, the problem in the thin-layer approximation is formulated as follows: to find a solution $h$, $T$ of system (2.4), (2.7) in the region $\omega$ such that conditions (2.3), (2.5), and (2.8.) are satisfied. Below, a problem (2.3)–(2.8) will be called a coupled problem. Problem (2.1)–(2.3), which contains the single required function $h$, will be called an uncoupled problem. We note that by the physical meaning, the function $h(x_1, x_2)$ cannot take negative values in the region $\omega$.

The limits of applicability of the approximation considered are discussed in [1]. Let $g = \mathrm{const}$ be the acceleration of gravity and $d = (2\sigma_0/(\rho g))^{1/2}$ be the capillary constant. The effect of gravity on the equilibrium shape of a nonisothermal film is insignificant if $h \ll d \sim l$. This inequality provides the upper estimate for $h$. The lower estimate of the quantity $h$ is $h \gg \eta$, where $\eta$ is the characteristic thickness of the double electric layer. If the last inequality is satisfied, the action of the wedging pressure can be ignored.

We introduce the dimensionless parameter $m = \rho g\beta h^2/\varkappa$, where $\beta$ is the volumetric thermal-expansion coefficient of the liquid. For $m \ll 1$, the contribution of buoyancy forces to the formation of the film profile and the velocity field in it can be ignored. As an example, we consider a pure water film under reduced gravity ($g = 1$ cm/sec$^2$) at a temperature of about 298 K. In this case, $d = 12$ cm. If we set $h = 0.1$ cm, $l = 5$ cm, $\eta = 10^{-6}$ cm, the inequalities $\eta \ll h \ll l$ are satisfied as well as the inequality $m \ll 1$ (in this case, $m = 1.6 \cdot 10^{-5}$). Under normal gravity $d = 0.38$ cm, the required inequalities are satisfied if $l \sim 0.5$ cm and $h \sim 0.05$ cm

**3. Uncoupled Problem (General Case).** If $\gamma = 0$ in Eq. (2.1), the unique solution of problem (2.1)–(2.3) is $h = 1$. This statement is proved by multiplying Eq. (2.1) by $\Delta h$ and integrating the resulting equality over the region $\omega$ subject to boundary conditions (2.2) and (2.3).

We assume that the parameter $\gamma$ is sufficiently small. Then, it is reasonable to seek a solution of problem (2.1)–(2.3) in the form of the power-law series

$$h = 1 + \sum_{k=1}^{\infty} \gamma^k h_k(x_1, x_2). \tag{3.1}$$

The function $h_1$ is a solution of the boundary-value problem

$$\Delta\Delta h_1 = \Delta\theta, \qquad (x_1, x_2) \in \omega; \tag{3.2}$$

$$\frac{\partial h_1}{\partial n} = 0, \qquad \frac{\partial \Delta h_1}{\partial n} = \frac{\partial \theta}{\partial n}, \qquad (x_1, x_2) \in \partial\omega; \tag{3.3}$$

$$\int\limits_{\omega} h_1\,d\omega = 0. \tag{3.4}$$

The functions $h_k$ $(k = 2, 3, \ldots)$ are determined sequentially as solutions of the problems

$$\Delta\Delta h_k = -\nabla\cdot\Big(\sum_{i=1}^{k-1} h_i \nabla\Delta h_{k-i}\Big), \qquad (x_1, x_2) \in \omega; \tag{3.5}$$

$$\frac{\partial h_k}{\partial n} = 0, \qquad \frac{\partial \Delta h_k}{\partial n} = -\sum_{i=1}^{k-1} h_i \,\frac{\partial \Delta h_{k-i}}{\partial n}, \qquad (x_1, x_2) \in \partial\omega; \tag{3.6}$$

$$\int_\omega h_k \, d\omega = 0. \tag{3.7}$$

Next, it is assumed that the curve $\partial\omega$ belongs to the Hölder class $C^{4+\alpha}$ $(0 < \alpha < 1)$ and the function $\theta(x_1, x_2)$ to the Hölder class $C^{2+\alpha}(\bar\omega\,)$. This provides the classical solvability of problems (3.2)–(3.7). Problem (3.2), (3.3) reduces to the Neumann problem for the Poisson equation

$$\Delta h_1 = \theta - \bar\theta, \quad (x_1, x_2) \in \omega, \qquad \frac{\partial h_1}{\partial n} = 0, \quad (x_1, x_2) \in \partial\omega, \tag{3.8}$$

where $\bar\theta$ is the average value of the function $\theta$ in the region $\omega$. By virtue of the additional condition (3.4), the function $h_1$ is determined uniquely.

We note that each of problems (3.5), (3.6) is an analog of the Neumann problem for the inhomogeneous biharmonic equation. The necessary solvability condition for this problem is given by the equality

$$\int_{\partial\omega} \sum_{i=1}^{k-1} h_i \,\frac{\partial \Delta h_{k-i}}{\partial n}\, ds = 0 \qquad (k = 2, 3, \ldots). \tag{3.9}$$

For problem (3.2), (3.3), the solvability condition is satisfied automatically. It turns out that it is also satisfied for problems (3.5) and (3.6) for any $k = 2, 3, \ldots$. To prove this, it is sufficient to substitute expression (3.1) into the equality following from (2.2)

$$\int_{\partial\omega} h \,\frac{\partial \Delta h}{\partial n}\, ds = \gamma \int_{\partial\omega} \frac{\partial \theta}{\partial n}\, ds$$

and to equate the coefficients at all powers of the parameter $\gamma$ to zero using the second condition (3.6). If the solvability condition (3.9) and the earlier formulated smoothness conditions are satisfied, each of problems (3.5) and (3.6) has a solution $h_k \in C^{4+\alpha}(\bar\omega\,)$, which is determined to within an additive constant. The arbitrariness that arises in the solution is eliminated by using condition (3.7). The convergence of series (3.1) in the norm of the space $C^{4+\alpha}(\bar\omega\,)$ for sufficiently small $\gamma$ is proved by a standard method using Schauder estimates of the solutions of the biharmonic equation.

We consider an approximate solution of problem (2.1)–(2.3) for the case where the region $\omega$ is a unit circle and the function $\theta$ is a simple harmonic polynomial: $\theta = x_1 x_2$. Because $\theta$ is harmonic, Eq. (2.1) is homogeneous and the effect of thermocapillary forces on the deformation of the film is manifested only through the second boundary condition (2.2). Transforming to the polar coordinates $(r, \varphi)$ on the plane $x_1, x_2$ and solving problems (3.2)–(3.7) sequentially, we obtain

$$\begin{gathered} h_1 = \frac{1}{24}\,(r^4 - 2r^2)\sin 2\varphi, \\ h_2 = \frac{1}{384}\Big[-\frac{1}{48}\,r^8 + \frac{1}{9}\,r^6 - \frac{1}{4}\,r^2 + \frac{7}{90} + \Big(\frac{1}{30}\,r^8 - \frac{11}{50}\,r^6 + \frac{79}{300}\,r^4\Big)\cos 4\varphi\Big]. \end{gathered} \tag{3.10}$$

Estimating the maximum values of the modules of the functions $h_1$ and $h_2$ in the circle $r \leqslant 1$, we obtain $\max|h_1| = 1/24 \approx 4.17\cdot 10^{-2}$ and $\max|h_2| \approx 4.13\cdot 10^{-4}$. This gives hope that the approximate solution of problem (2.1)–(2.3) $h^{(3)} = 1 + \gamma h_1 + \gamma^2 h_2$ approximates its exact solution well, at least, for values of the parameter $\gamma$ of the order of unity.

We consider a pure water film 0.05 cm thick and 0.5 cm in diameter at room temperature. In this case, the value $\gamma = 1$ corresponds to a temperature drop at its edges of $\delta T = 3.64$ K. Formulas (3.10) show that for the given temperature distribution $\theta = (r^2 \sin 2\varphi)/2$ and the values of $\gamma \sim 1$, considerable deformations of the film profile are concentrated near its boundaries. Thus, if $\gamma = 2$, the deviation of the film thickness in the circle $r \leqslant 0.8$ from its average value $\bar{h} = 1$ does not exceed 0.08. It is of interest to calculate the expression $\gamma\Delta(h_1 + \gamma h_2) = H^{(2)}$, which, in the thin-layer approximation, approximates the average film curvature. According to (3.1) and (3.10),

$$H^{(2)} = \frac{\gamma}{2} r^2 \sin 2\varphi + \frac{\gamma^2}{48}\Big[-\frac{1}{6} r^6 + \frac{1}{2} r^4 - \frac{1}{16} + \Big(\frac{1}{5} r^6 - \frac{11}{20} r^4\Big)\cos 4\varphi\Big].$$

From this it follows that the largest values of the function $|H^{(2)}(r, \varphi)|$ are reached on the circle $r = 1$. If $\gamma \leqslant 2$, then $\max |H^{(2)}| < 1.023$.

We note that for sufficiently small values of $\gamma$, the solution $h(x_1, x_2)$ of problem (2.1)–(2.3) is positive in the closed region $\bar{\omega}$. As $\gamma$ increases, this property can be lost. The one-dimensional versions of the problem corresponding to the cases where $\omega$ is a circle and the function $h$ is radially symmetric or $\omega$ is a strip $|x_1| <$ const (in this case, $\omega$ does not depend on $x_2$) were studied in [1]. It was shown that for $4\theta = -(x_1^2 + x_2^2) \equiv -r^2$ (or $2\theta = -x_1^2$) there exists $\gamma^*$ such that $h = 0$ at the point $r = 0$ (or $x_1 = 0$) whereas in the remaining part of the region $\omega$, we have $h > 0$. It was established that in the axisymmetric case, $\gamma^* \approx 32.4$ and in the plane case, $\gamma^* \approx 39.2$. Approximate solution of problem (2.1)–(2.3) in the case where $\omega$ is a unit circle and $2\theta = r^2 \sin 2\varphi$ provides a rough estimate of the values of the parameter $\gamma$ for which a positive solution of this problem still exists: $\gamma < 24$. The following hypothesis seems plausible. Let $\Delta\theta = -1$ in the region $\omega$ and $\partial\theta/\partial n < 0$ on its boundary. Then, there exists a value $\gamma^* > 0$ such that for $\gamma > \gamma^*$ there is no positive solution of problem (2.1)–(2.3). Proving this hypothesis is a rather complicated problem of the theory of degenerating quasilinear elliptic equations.

**4. Uncoupled Problem (Axisymmetric Case).** We assume that $\omega$ is a circle and $\theta = -r^2/4$. We seek axisymmetric solutions of problem (2.1)–(2.3) $h = h(r)$. Then, Eq. (2.1) admits single integration. Taking into account the second boundary condition (2.2), we obtain the following problem:

$$h \frac{d}{dr}\Big[\frac{1}{r}\frac{d}{dr}\Big(r\frac{dh}{dr}\Big)\Big] = -\gamma r, \qquad 0 < r < 1; \tag{4.1}$$

$$h \quad \text{and} \quad \frac{d^2 h}{dr^2} \quad \text{are bounded,} \qquad \frac{dh}{dr} \to 0 \quad \text{as} \quad r \to 0; \tag{4.2}$$

$$\frac{dh}{dr} = 0 \quad \text{for} \quad r = 1; \tag{4.3}$$

$$\int_0^1 r h(r)\, dr = \frac{1}{2}. \tag{4.4}$$

The plane analog of problem (4.1)–(4.4) is written as

$$h \frac{d^3 h}{dx^3} = -\gamma x, \qquad 0 < x < 1; \tag{4.5}$$

$$\frac{dh}{dx} = 0 \quad \text{for} \quad x = 0 \quad \text{and} \quad x = 1; \tag{4.6}$$

$$\int_0^1 h(x)\, dx = 1. \tag{4.7}$$

Equation (4.4) can be integrated again:

$$h \frac{d^2 h}{dx^2} - \frac{1}{2}\Big(\frac{dh}{dx}\Big)^2 = -\frac{1}{2}\gamma x^2 + \zeta. \tag{4.8}$$

The integration constant $\zeta$ is a functional of the solution of problem (4.5)–(4.7). In [1], it is proved that problem (4.5)–(4.7) has at least one positive solution if $0 \leqslant \gamma < 9$. If $\gamma < 9(1 - \pi^{-3/2})^2$, this solution is unique. For sufficiently large values of $\gamma$, the problem has no positive solutions. Although the upper estimate of the admissible range of the parameter $\gamma$ found in [1] is five orders of magnitude larger than the real value obtained numerically:

$0 \leqslant \gamma < \gamma^* \approx 39.2$. This result is of fundamental importance. Below, the same result is obtained for the solutions of problem (4.1)–(4.4).

In relations (4.1)–(4.4), we transform to the new independent variable $t = r^2$ and the new required function $u(t) = h(r)$. The function $u$ is a solution of the following boundary-value problem:

$$u(tu')'' = -a, \qquad 0 < t < 1; \tag{4.9}$$

$$u,\ u',\ u'' \quad \text{are bounded} \quad \text{as} \quad t \to 0; \tag{4.10}$$

$$u' = 0, \qquad t = 1; \tag{4.11}$$

$$\int\limits_0^1 u\,dt = 1. \tag{4.12}$$

Here $a = \gamma/4$; the prime denotes differentiation with respect to $t$. A solution of Eq. (4.9) that satisfies conditions (4.10) will be called a regular solution. It is required to obtain *a priori* estimates of the regular positive solution of problem (4.9)–(4.12) for $a > 0$.

We denote $u'' = p$. The function $p(t)$ satisfies the equation $tp' + 2p = -au^{-1}$. Let $u$ be a regular positive solution of problem (4.9)–(4.12). Then, for the solution of the last equation, which is bounded as $t \to 0$, the following representation is valid:

$$p(t) = -\frac{a}{t^2}\int\limits_0^t \frac{x\,dx}{u(x)}.$$

This implies that $u'' < 0$ for all $t \in [0, 1]$. Integrating the last equality from $t$ to 1 and using condition (4.11), we obtain the representation

$$u'(t) = a\int\limits_t^1 \Bigl(\int\limits_0^s \frac{x\,dx}{u(x)}\Bigr)\frac{ds}{s^2}.$$

From this it follows that for any regular positive solution $u(t)$ of problem (4.9)–(4.12), the inequality $u' > 0$ holds if $0 \leqslant t < 1$.

We denote $u''' = q$. Dividing Eq. (4.9) by $u$ and differentiating the resulting equality, we obtain the equation $tq' + 3q = au^{-2}u'$. We note that for any regular solution of Eq. (4.9) the function $tu'''(t)$ has a finite limit as $t \to 0$. This allows us to obtain the following representation for the function $q$ by integrating the last equation from $t = 0$ to the current value of $t$:

$$q(t) = \frac{a}{t^3}\int\limits_0^t \frac{u'(x)x^2\,dx}{u(x)}.$$

By virtue of the positiveness of $u$ and $u'$ for $t \in [0, 1)$ and the definition of the function $q$, it can be concluded that $u'''(t) > 0$ for all $t \in [0, 1]$.

Next, *a priori* estimates of the solution of problem (4.9)–(4.12) are needed. The simplest of them is obtained by integrating Eq. (4.9) from 0 to 1 subject to boundary conditions (4.10) and (4.11):

$$u(0)u'(0) - u(1)u''(1) + \frac{1}{2}\int\limits_0^1 [u'(t)]^2\,dt = a. \tag{4.13}$$

We note that $u(1)u'''(1) + 2u(1)u''(1) = -a$ by virtue of Eq. (4.9). Since $u'''(1) > 0$, this implies the inequality

$$u(1)u''(1) < -a/2. \tag{4.14}$$

Taking into account that $u(0)u'(0)$ is positive, from (4.13) and (4.14), we obtain the estimate

$$\int\limits_0^1 [u'(t)]^2\,dt < a. \tag{4.15}$$

Inequality (4.15) provides bilateral estimates of the function $u$. Indeed, by virtue of condition (4.12), a point $t_1 \in [0,1]$ exists such that $u(t_1) = 1$. This allows the function $u$ to be represented as

$$u(t) = 1 + \int_{t_1}^{t} u'(x)\,dx.$$

This relation and Eq. (4.15) lead to the estimates $1 - [a|t - t_1|]^{1/2} < u(t) < 1 + [a|t - t_1|]^{1/2}$.

Strengthening the upper estimate, we find that for $a > 0$,

$$u(t) < 1 + a^{1/2}, \qquad 0 \leqslant t \leqslant 1. \tag{4.16}$$

The lower estimate can also be strengthened:

$$u(t) > 1 - a^{1/2}, \qquad 0 \leqslant t \leqslant 1, \tag{4.17}$$

but, unlike (4.16), this estimate is informative only for $a < 1$. Nevertheless, it allows one to prove the solvability of problem (4.1)–(4.4) for small values of the parameter $\gamma = 4a$. Such a proof is given in [1] but the guaranteed interval of existence of the solution of the problem is not indicated. Based on estimate (4.17), it can be argued that a positive solution of problem (4.1)–(4.4) necessarily exists if $\gamma < 4$. For even smaller values of $\gamma$, the positive solution is unique (a proof of this fact is not given here).

We prove that for large values of $a$, problem (4.9)–(4.12) has no positive solutions. In contrast to Eq. (4.5), Eq. (4.9) does not admit integration. Therefore, the approach proposed in [1] cannot be used directly to prove the failure of the solution of problem (4.5)–(4.7) for sufficiently large big $\gamma$. However, the certain similarity between both problems suggests the correct line of reasoning. [We note that the order of Eq. (4.9) can be reduced because it is invariant under the stretching transformation $\tilde{t} = ct$, $\tilde{u} = c^2 u$, but this circumstance does not help obtain the desired result.]

In Eq. (4.9), we transform to the new required function $v = u^{1/2}$. The function $v(t)$ satisfies the equation

$$v'' = \frac{f(t)}{2tv^3} - \frac{v'}{tv^2}, \tag{4.18}$$

where

$$f(t) = -at + a + u(1)u''(1) - \frac{1}{2}\int_t^1 [u'(x)]^2\,dx. \tag{4.19}$$

Using representation (4.19) and inequality (4.14), it can be concluded that the function $f$ admits the upper estimate

$$f(t) < -a(t - 1/2), \qquad 0 \leqslant t \leqslant 1. \tag{4.20}$$

In this case, it is of significance that for $t > 1/2$, the function $f$ takes negative values.

We integrate Eq. (4.18) from $t \geqslant 1/2$ to 1, taking into account that $v'(1) = 0$ by virtue of condition (4.11) and the definition of $v$. As a result, we obtain

$$v'(t) = \int_t^1 \Big( -\frac{f(x)}{2xv^3(x)} + \frac{v'(x)}{xv^2(x)} \Big)\,dx. \tag{4.21}$$

The further reasoning is based on finding the lower estimate of the function $v$ that, for large values of $a$, leads to a contradiction with condition (4.12). For this purpose, we estimate the integrand in (4.21) from below using inequality (4.20), $v' = 2uu' > 0$, and the inequality $v < (1 + a^{1/2})^{1/2}$, which follows from (4.16). As a result, we find

$$v'(t) > \frac{a}{2(1 + a^{1/2})^{3/2}} \int_t^1 \Big(1 - \frac{1}{2x}\Big)\,dx = \frac{a}{2(1 + a^{1/2})^{3/2}} \Big(1 - t + \frac{1}{2}\ln t\Big), \qquad \frac{1}{2} \leqslant t \leqslant 1.$$

Integration of the obtained inequality from $t = 1/2$ to the current value of $t$ gives the desired estimate

$$v(t) > \frac{a\eta(t)}{2(1 + a^{1/2})^{3/2}} + v\Big(\frac{1}{2}\Big), \qquad \frac{1}{2} \leqslant t \leqslant 1, \tag{4.22}$$

where

$$\eta(t) = \frac{1}{2}\Big(t - t^2 + t\ln t + \frac{1}{2}\,\ln 2 - \frac{1}{4}\Big).$$

The function $\eta(t)$ possesses the following properties: 1) $\eta(1/2) = 0$; 2) $\eta(t)$ increases strictly for $t \in (1/2, 1]$. This implies that

$$\int\limits_{1/2}^{1} \eta^2(t)\,dt = C > 0.$$

Using inequality (4.22) and taking into account the definition $v = u^{1/2}$ and the positiveness of $v(1/2)$, it can be concluded that

$$\int\limits_{1/2}^{1} u(t)\,dt > \frac{C^2a^2}{4(1 + a^{1/2})^3}.$$

It is obvious that for the positive solutions of problem (4.9)–(4.12), the last inequality contradicts condition (4.12) if $a$ is sufficiently large. This implies that for large values of the parameter $\gamma = 4a$, the original axisymmetric problem has no positive solutions.

**5. Coupled Problem (General Case).** This problem consists of determining a pair of functions $h$ and $T$ that satisfy Eqs. (2.4) and (2.7) in the region $\omega \in \mathbb{R}^2$, boundary conditions (2.5) and (2.8) on its boundary $\partial\omega$, and the additional condition (2.3). The controlling functional parameter of the problem is the function $q$ that specifies the heat flux distribution on the curve $\partial\omega$. This function is subject to the necessary condition (2.6). In this case, the function $T$ in the solution of problem (2.3)–(2.8) is not uniquely determined. The arbitrariness in its determination can be eliminated by requiring that the following condition be satisfied:

$$\int\limits_{\omega} T(x_1, x_2)\,d\omega = 0. \tag{5.1}$$

Next, it is assumed that condition (5.1) is satisfied.

By analogy with the uncoupled problem (2.1)–(2.3)–(2.8), the solution of problem (2.3) can be sought in the form

$$h = 1 + \sum_{k=1}^{\infty} \gamma^k h_k(x_1, x_2), \qquad T = \sum_{k=0}^{\infty} \gamma^k T_k(x_1, x_2). \tag{5.2}$$

The functions $T_0$ and $h_1$ constitute the solution of the following problem:

$$\Delta T_0 = 0, \qquad \Delta\Delta h_1 = 0, \qquad (x_1, x_2) \in \partial\omega; \tag{5.3}$$

$$\frac{\partial T_0}{\partial n} = q, \qquad \frac{\partial h_1}{\partial n} = 0, \qquad \frac{\partial \Delta h_1}{\partial n} = q, \qquad (x_1, x_2) \in \partial\omega; \tag{5.4}$$

$$\int\limits_{\omega} T_0\,d\omega = 0, \qquad \int\limits_{\omega} h_1\,d\omega = 0. \tag{5.5}$$

The functions $T_k$ and $h_{k+1}$ ($k = 1, 2, \dots$) are found from the recursive system of equations and boundary conditions:

$$\begin{gathered} \Delta T_k = -\nabla\cdot\Big(\sum_{i=1}^{k} h_i \nabla T_{k-i}\Big), \\ \Delta\Delta h_{k+1} = \Delta T_k - \nabla\cdot\Big(\sum_{i=1}^{k} h_i \nabla\Delta h_{k+1-i}\Big), \qquad (x_1, x_2) \in \omega; \end{gathered} \tag{5.6}$$

$$\frac{\partial T_k}{\partial n} = -\sum_{i=1}^{k} h_i\,\frac{\partial T_{k-i}}{\partial n}, \qquad \frac{\partial h_{k+1}}{\partial n} = 0,$$

$$\frac{\partial \Delta h_{k+1}}{\partial n} = \frac{\partial T_k}{\partial n} - \sum_{i=1}^{k} h_i \frac{\partial \Delta h_{k+1-i}}{\partial n}, \qquad (x_1, x_2) \in \partial\omega; \tag{5.7}$$

$$\int\limits_{\omega} T_k \, d\omega = 0, \qquad \int\limits_{\omega} h_{k+1} \, d\omega = 0. \tag{5.8}$$

We assume that the curve $\partial\omega$ belongs to the Hölder class $C^{4+\alpha}$ $(0 < \alpha < 1)$ and the function $q$ to the Hölder class $C^{1+\alpha}(\partial\omega)$. Then each of problems (5.3)–(5.5) and (5.6)–(5.8) has a unique solution $T_k \in C^{2+\alpha}(\bar{\omega})$, $h_{k+1} \in C^{4+\alpha}(\bar{\omega})$ $(k = 0, 1, 2, \ldots)$. If the parameter $\gamma > 0$ is sufficiently small, series (5.2) converge, in the norms of the spaces $C^{4+\alpha}(\bar{\omega})$ and $C^{2+\alpha}(\bar{\omega})$, to the functions $h$ and $T$, respectively, that constitute the solution of problem (2.3)–(2.8).

We note that the function $T_0$ is harmonic. This allows us to compare the solution of the uncoupled problem considered in Sec. 3 with the solution of the coupled problem in the case where $\omega$ is a unit circle and $q = \sin 2\varphi$ (we recall that $r$ and $\varphi$ are polar coordinates). With this specification of the function $q$, the equality $\theta = T_0$ is satisfied. The coincidence of the functions $\theta$ and $T_0$ is responsible for the equality of the functions $h_1$ in the solutions of both problems. However, because of the film deformation, the next terms of the temperature expansion (5.2) are not harmonic functions. Calculations show that in the case $q = \sin 2\varphi$

$$T_1 = \frac{1}{288} r^6 + \frac{1}{96} r^4 - \frac{1}{384} + \Big(\frac{1}{480} r^6 - \frac{1}{30} r^4\Big) \cos 4\varphi.$$

We note that the functions $h_1$ and $T_0$ constituting the solution of problem (5.3)–(5.5) are linked by the relation $\Delta h_1 = T_0$. From this it follows that the function $h_2$ obtained by solving problem (5.6)–(5.8) for $k = 1$ differs from the function $h_2$ defined by the second formula (3.10) only by a factor of 2.

**6. Coupled Problem (Plane Case).** Below, we study the solutions of system (2.4), (2.7) in which the functions $h$ and $T$ depend only on one variable $x_1 = x$. These solutions describe the equilibrium of a nonisothermal film bounded by the planes $x = 0$ and $x = 1$ provided that its free surface is thermally insulated and that a heat flux with a constant dimensionless density $q$ is specified on the solid boundaries. In this case, Eqs. (2.4) and (2.7) are integrated to yield the system

$$h\dddot{h} = \gamma \dot{T} + d, \qquad h\dot{T} = q,$$

where the dot denotes differentiation with respect to $x$ and $d$ is constant. By virtue of the second boundary condition (2.8), $d = 0$. Eliminating the function $\dot{T}$ from the relations obtained, we arrive at the following equation for the film thickness $h$:

$$h\dddot{h} = -bh^{-1}, \qquad 0 < x < 1, \tag{6.1}$$

where $b = -\gamma q$. Boundary conditions (2.3) and the first of conditions (2.8) in the one-dimensional case become

$$\dot{h}(0) = \dot{h}(1) = 0; \tag{6.2}$$

$$\int\limits_0^1 h(x) \, dx = 1. \tag{6.3}$$

We note that without loss of generality, the number $b$ can be considered nonnegative. The case $b < 0$ is reduced to the case $b > 0$ by the substitution $\tilde{x} = 1 - x$.

If $b = 0$, the unique solution of problem (6.1)–(6.3) is $h = 1$. For small $b$, its solution has the asymptotic form

$$h = 1 + b\Big(-\frac{1}{24} + \frac{1}{4} x^2 - \frac{1}{6} x^3\Big) + O(b^2), \qquad b \to 0. \tag{6.4}$$

From (6.4) it follows that for small $b$, the function $h(x)$ increases strictly monotonically and has a unique point of inflection. These qualitative properties of the solution are retained for large values of the parameter $b$. The existence of more than one point of inflection is impossible because $\dddot{h} < 0$ for positive solutions of problem (6.1)–(6.3). The same inequality implies that the function $\ddot{h}(x)$ decreases strictly. If $\ddot{h}(0) \leqslant 0$, the condition $\dot{h}(1) = 0$ cannot be satisfied. Thus, $\ddot{h}(0) > 0$, which leads to a monotonic increase in the function $h(x)$ for any value $b > 0$.

The positiveness of $b$ implies that the temperature decreases with increasing $x$ whereas the film thickness increases. This is natural since the surface-tension coefficient (1.1) increases as the temperature decreases. In addition, the film thickness at the heated end $h(0)$ decreases with increasing parameter $b$. For small values of $b$, this follows from formula (6.4). Is of interest to elucidate whether for the positive solution $h(x)$ of problem (6.1)–(6.3), the quantity $h(0)$ can vanish for any value of the parameter $b > 0$. For this, we study the behavior of the solutions of Eq. (6.1) such that $h(x) \to 0$ as $x \to 0$.

In Eq. (6.1), we transform to the new independent variable $s$ and the new required function $z$ using the formulas

$$h = \exp s, \qquad \dot{h} = z(s). \tag{6.5}$$

The function $z(s)$ satisfies the second-order equation

$$z^2\Big(\frac{d^2z}{ds^2} - \frac{dz}{ds}\Big) + z\Big(\frac{dz}{ds}\Big)^2 = -b. \tag{6.6}$$

For Eq. (6.6), we consider the Cauchy problem

$$z = c, \qquad \frac{dz}{ds} = 0 \qquad \text{at} \qquad s = s_*, \tag{6.7}$$

where $s_* = \log h(x_*)$; $c = \dot{h}(x_*)$, and $x_*$ is the value of $x$ for which the function $\dot{h}(x_*)$ takes the largest value. We note that for $b > 0$, this value is unique and $0 < x_* < 1$. In addition, $c > 0$. The substitution

$$\Big(z\,\frac{dz}{ds}\Big)^2 = w(z) \tag{6.8}$$

reduces Eq. (6.6) to the first-order equation

$$\Big(\frac{dw}{dz} + 2\Big)^2 = z^2 w, \tag{6.9}$$

which is equivalent to the two equations

$$\frac{dw_1}{dz} = -2 - z w_1^{1/2}; \tag{6.10}$$

$$\frac{dw_2}{dz} = -2 + z w_2^{1/2}. \tag{6.11}$$

The functions $w_1$ and $w_2$ cannot take negative values by virtue of their definition. For Eqs. (6.12) and (6.13), we consider the Cauchy problems

$$w_k(c) = 0, \qquad k = 1, 2. \tag{6.12}$$

The right sides of Eqs. (6.10), (6.11) lose smoothness at the point $z = c$, $w_k = 0$. However, both Cauchy problems (6.10), (6.12) and (6.11), (6.12) have unique solutions defined in the left half-neighborhood of the point $z = c$. Of interest is the solution of the first of the indicted problems.

The integral curve of Eq. (6.10) that leaves the point $z = c$, $w_1 = 0$ is above the straight line $w_1 = 2b(c - z)$. Since the right side of this equation increases sublinearly along the variable $w_1$, the solution of the Cauchy problem (6.10), (6.12) can be continued to the point $z = 0$; $w_1(0) > 2bc$ if $b > 0$, and the following inequality holds:

$$w_1 > 2b(c - z), \qquad 0 < z < c. \tag{6.13}$$

If the solution of problem (6.10), (6.12) is known, the function $s_1(z)$ can be defined by the relation

$$s_1(z) = -\int\limits_z^c \frac{\zeta\, d\zeta}{[w_1(\zeta)]^{1/2}} + s_*, \qquad 0 \leqslant z \leqslant c. \tag{6.14}$$

Knowledge of this function allows us to find the parametric dependence $h(x)$ on the interval $0 \leqslant x \leqslant x_*$ using relations (6.5) and (6.8). By virtue of (6.5), (6.13), and (6.14), the following *a priori* estimate is valid:

$$h(0) > h(x_*) \exp\{-2^{3/2} 3^{-1} b^{-1/2} [\dot{h}(x_*)]^{3/2}\}, \qquad b > 0. \tag{6.15}$$

From this estimate, it follows that for any finite value of $b > 0$, the quantity $h(0)$ is positive. Violation of inequality (6.15) implies that $\dot{h}(x_*) \to \infty$ as $x \to x_*$ but this is impossible by virtue of Eq. (6.1) since $h(x_*) > 0$.

Estimate (6.15) yet does not allow one to prove that problem (6.1)–(6.3) is solvable for any value $b > 0$ but it may be useful in the numerical solution by continuation over the parameter $b$. The question of the solvability of problem (6.1)–(6.3) for all positive $b$ needs to be further investigated, both analytically and numerically.

In conclusion, we emphasize again that in the present paper, by the equilibrium of a free nonisothermal film is meant its stationary shape and the stationary velocity and temperature fields in the film. In the thin-layer approximation, these two problems (determining the equilibrium film shape and finding the velocity and temperature fields) are solved sequentially. The present paper is devoted to the first of these problems. The second problem is formulated in [1], where its approximate solution is given for small values of the Marangoni number provided that the film shape is known.

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